Using the standard recursion relation,

$$(n-m)P_n^m = (2n-1)P_{n-1}^m - (n+m-1)P_{n-2}^m$$

and the fact that $P_n^m = 0$ for $m > n$, a simple relation can be obtained for P_{n+1}^n:

$$P_{n+1}^n = (2n+1)P_n^n \tag{5}$$

Then, Eqs. (4) and (5) can be used to compute any P_n^m given the initial value $P_1^1 = (1-x^2)^{1/2}$.

Alternative 2

An alternative mechanization of Eq. (3) is

$$P_n^{m-1} = \frac{m\alpha P_n^m - P_n^{m+1}}{(n+m)(n-m+1)} \tag{6}$$

In this case, we first compute all P_n^n up to $n = N$ using Eq. (4) with the starting value $P_1^1 = (1-x^2)^{1/2}$. Equation (6) is then solved starting with P_n^n and proceeding with decreasing m to $m = 1$ for given n. Where appropriate, we again make use of the relationship $P_n^m = 0$ for $m > n$ in utilizing Eq. (6). In contrast to Eq. (3) applied in the forward sense, Eq. (6) is stable when run backward over m. An analogous instability has been discussed for Bessel functions.[4]

Conclusions

Use of the standard recursion relationships for the associated Legendre functions will, in general, lead to numerical roundoff errors in the computed gravity. These roundoff errors can avalanche catastrophically near the poles unless the recursion relations used are stable. Two examples of stable recursion relations have been discussed and an unstable one analyzed.

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The Minimum for Geometric Dilution of Precision in Global Positioning System Navigation

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In a previous Note,[1] the author presented some simple bounds to the global positioning system (GPS) navigation performance index, the geometric dilution of precision (GDOP). It was shown that the GDOP must be greater than $\sqrt{2}$ and that a value as low as $\sqrt{2.5}$ was attained for a completely symmetrical GPS configuration; i.e., the line-of-sight vectors from the user to the four GPS satellites were all separated by the same angle $\cos^{-1}(-1/3)$. It will be shown below that the value of $\sqrt{2.5}$ is indeed the minimum for GDOP.

As shown in Ref. 1,

$$\mathrm{GDOP} = \sqrt{1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 + 1/\lambda_4} \tag{1}$$

where the λ are eigenvalues of a 4×4 real symmetric nonnegative matrix with a trace equal to 8. The results of Ref. 1 were obtained by examining the 4×4 matrix HH^T, but it was pointed out that this 4×4 matrix may also be

$$H^TH = \begin{bmatrix} aa^T + bb^T + cc^T + dd^T & a+b+c+d \\ (a+b+c+d)^T & 4 \end{bmatrix} \tag{2}$$

where

$$H^T = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is the measurement partial derivative matrix, transposed; a, b, c, and d are line-of-sight unit vectors from a set of four GPS satellites to the user. From the well-known mini-max property of the eigenvalues of symmetric matrices,[2] one has, for the largest eigenvalue λ_4 of H^TH,

$$\lambda_4 \geq [0\ 0\ 0\ 1]\,(H^TH) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 4 \tag{3}$$

Subject to this constraint and the fact that all the λ are nonnegative and have a sum equal to 8, one has

$$\mathrm{GDOP} \geq \mathrm{minimum}\sqrt{[1/\lambda_4 + 3/[(8-\lambda_4)/3]} = \sqrt{2.5}$$

occurring at $\lambda_4 = 4$. That 2.5 is a minimum and not a lower bound has already been shown by the construction of the completely symmetrical GPS configuration.

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Optimization of Cruise at Constant Altitude

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Nomenclature

C = cost function
C_{D_0} = drag coefficient at zero lift

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C_{L_α} = lift coefficient slope, rad^{-1}
c_t = cost of flight time, s^{-1}
c_w = cost of fuel, lb^{-1}
D = drag, lb
F_1 = cruise cost function, type 1
F_2 = cruise cost function, type 2
F_3 = cruise cost function, type 3
f = fuel flow, $lb \cdot s^{-1}$
H = Hamiltonian
H_0 = optimum value of H
L = lift, lb
S = aerodynamic reference area, ft^2
t_f = final time, s
v = true airspeed, $ft \cdot s^{-1}$
w = weight, lb
w_f = final weight, lb
x = range, ft
α = incidence angle, rad
λ_x = range adjoint
λ_w = weight adjoint
ρ = air density, $slug \cdot ft^{-3}$

Introduction

INCREASES in fuel costs have prompted the development of fuel savings algorithms to control the climb cruise and descent phases of aircraft flight. The more sophisticated methods involve extensive in-flight computations to adjust the thrust and airspeed on the basis of aircraft design parameters and the prevailing atmospheric conditions. An algorithm of this kind, developed by Erzberger et al., has gained wide acceptance and forms the basis for fuel savings systems on many commercial aircraft. However, it involves an approximation in the cruise cost function that can introduce significant errors. In this Note, the nature of the approximation is discussed and some numerical estimates of the errors involved are made.

Analysis

When aircraft flight is restricted to a fixed altitude in still air, the steady-state equations of motion can be written as

$$\dot{x} = v, \quad \dot{w} = -f$$

where x and w are the state variables distance and weight and v and f the airspeed and fuel flow. At constant altitude, lift and weight are equal and, in the steady-state approximation, thrust is assumed to be equal to drag. These conditions make it possible to compute the thrust required to maintain a given airspeed and so to express the fuel flow f in terms of airspeed v, which is taken as the control variable. The objective, then, is to adjust the airspeed throughout the flight in such a way as to consume the minimum amount of fuel, that is, to arrive at the destination with maximum weight. More generally, we may wish to maximize the final value of a cost function defined as

$$C = c_w w_f - c_t t_f$$

where c_w is the cost of fuel per pound and c_t the cost of flight time per second. Minimum fuel is then the special case of $c_t = 0$.

An explicit method for solving this problem is provided by the theory of optimization.[1] The Hamiltonian is defined as

$$H = \lambda_x v - \lambda_w f$$

where λ_x and λ_w are the costate or adjoint variables, which are functions of time defined by the differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} \quad \text{and} \quad \dot{\lambda}_w = -\frac{\partial H}{\partial w}$$

and the optimum airspeed at any time is that providing a stationary value for H, that is, the speed at which $\partial H / \partial v = 0$. These four differential equations for x, w, λ_x, and λ_w require four boundary conditions. For flight with a fixed range, but final time open, we have: at $t = t_0$, $x = x_0$ and $w = w_0$; at $x = x_f$, $\lambda_w = c_w$ and $H = c_t$. All that is required to provide a solution is a numerical integration procedure plus an iteration method to handle the two-point boundary values, both of which are readily available.

There are, however, further nuances to the problem that are of some interest. We note first that H is not an explicit function of range and so λ_x is constant. Also, since H is not an explicit function of time H itself takes a constant value H_0 along the optimum flight path.

Instead of finding the stationary point of H, we may use an equivalent function with a stationary point at the same value of v if this is more convenient. An example of such a function is that derived by Burrows[2] with slightly different notation,

$$F_1 = (c_t + \lambda_w f)/v$$

A second such equivalent function,[3]

$$F_2 = f/(v - c_t/\lambda_x)$$

is of interest since it contains only f and v and the constant c_t/λ_x, thus avoiding the rather laborious computation of λ_w. It is currently used in the fuel savings algorithm for the KC-135 fleet of air tankers.

A method for flight path optimization has been proposed by Erzberger et al.,[4,5] which forms the basis for many operational fuel savings systems. It includes methods for flying the climb and descent phases and for selecting the cruise altitude, but the validity of the system is dependent on the correct choice of the cruise cost function. During cruise, the airspeed is determined by minimizing a cruise cost function defined as

$$F_3 = (c_t + c_w f)/v$$

Table 1 Fixed cost flight paths

Range, n.mi.	v_{opt} ft/s	v_{F_3} ft/s
0	842.1	822.5
500	812.4	794.1
1000	782.6	765.8
1500	752.9	737.7
2000	723.3	709.7
2500	693.7	682.0
3000	664.2	654.6
3500	634.8	627.4
4000	605.5	600.6
4500	576.3	574.2
5000	547.3	548.3

Table 2 Four-dimensional flight paths

Range, n.mi.	v_{opt} ft/s	v_{F_3} ft/s
0	642.7	689.3
500	614.7	655.4
1000	586.6	621.0
1500	558.6	586.3
2000	530.7	551.0
2500	502.9	515.3
3000	475.1	479.0
3500	447.3	441.9
4000	419.7	403.9
4500	392.1	364.7
5000	364.6	324.0

This function does not provide the same value for optimum v as does the Hamiltonian or the equivalent functions, F_1 and F_2. Comparing F_3 with F_1, it will be recognized as the approximation in which λ_w is replaced by its final value c_w and the error introduced therefore depends on the extent to which λ_w differs from c_w during the earlier part of the flight.

Some indication of the extent to which the airspeed computed from F_3 deviates from optimum is provided by two numerical examples using an aircraft with idealized aerodynamic and propulsive characteristics. The lift, assumed equal to weight, is given by

$$L = \tfrac{1}{2}\rho v^2 S C_{L_\alpha} \alpha$$

and the drag, assumed equal to thrust, is given by

$$D = \tfrac{1}{2}\rho v^2 S (C_{D_0} + C_{L_\alpha} \alpha^2)$$

where ρ is the air density, 0.001267 slugs/ft^3 at 20,000 ft; S the reference area, 3500 ft^2; and the aerodynamic coefficients $C_{L_\alpha} = 10$ and $C_{D_0} = 0.015$. The fuel flow is given by $f = kT$, where k has the value 0.8 lb/h/lb$_{\text{thrust}}$. For a given weight and airspeed, the first of these two equations defines the required angle of incidence α and substitution into the second gives the thrust T required to maintain that airspeed. Multiplication by k then gives the fuel flow. These equations, for a given value of weight, define the function $f(v)$. In the examples given here, the differential equations were integrated forward using a Runge-Kutta method, with iteration procedures to satisfy the final boundary conditions.

With an initial weight of 300,000 lb and cost factors of \$1/lb of fuel and \$1/s of flight time, the optimum velocity profile for a range of 5000 mi. is given in the second column of Table 1. The optimum flight time and final weight are 44,456 s and 109,926 lb. The velocity profile computed from F_3, given in the third column, shows a significant deviation from optimum in the early part of the flight. The flight time and weight in this case are 45,087 s and 110,493 lb, for an additional cost of \$64 with respect to the optimum.

In the case where fuel usage is to be minimized with both range and flight time specified, sometimes referred to as four-dimensional navigation, the same differential equations are integrated but with the boundary condition, $H = c_t$, replaced by the terminal condition, $t = t_f$. The same cruise cost function F_3 has been proposed by Sorensen and Waters[6] and Burrows[7] to compute such trajectories, now using c_t as a parameter to adjust the time of flight. As may be expected, this provides suboptimal flight paths. An example is given in Table 2 for the same aircraft flying over the same range, but with the flight time specified as 62,000 s. The computed velocity profile given in the third column is quite different from the optimum profile in the second column and the fuel usage is 621 lb more than optimum, representing a very significant 0.31% loss. Further details of these computations can be made available on request to the authors.

Conclusions

Cruise trajectories computed by means of the cruise cost function proposed by Erzberger and others and, therefore, also the global flight paths of which they are a part, have been found to be suboptimal. The extent to which the solutions deviate from the optimum may be small in many cases, but an example has been given in which the deviation is quite significant, amounting to a fuel loss of 0.3%. This is an extreme case with an extended flight time unlikely to be used in commercial operations, but it serves to indicate that significant errors can arise and that caution is needed in applying the approximate method, particularly in four-dimensional applications. It is suggested that, unless the error can be estimated or at least bounded for each trajectory, an exact cruise cost function should be used, possibly selected from those discussed above.

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Multivariable Control Robustness Examples: A Classical Approach

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Introduction

THIS Note discusses two examples that have been used to demonstrate the superiority of multivariable control analysis/synthesis techniques based on singular values over classical techniques. Classical stability analysis, by breaking loops one at a time, is known to be an unreliable way of testing robustness to simultaneous perturbations of all the loops. However, classical stability analysis, by successively closing loops, is more reliable. The difference between these two classical approaches has not been well recognized. In this Note, we show that the successive-loop-closure way of analyzing these two examples does predict a lack of robustness in the nominal designs and, furthermore, provides insights into how the design can be changed so as to be more robust.

Examples of Multivariable Control Robustness

Two examples that have been used for the discussion of multivariable control loop robustness are:

Example 1 (Refs. 1, 2):

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{s^2+100} \begin{bmatrix} s-100 & -10(s+1) \\ -10(s+1) & s-100 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{1}$$

Example 2 (Ref. 3):

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} -47s+2 & 56s \\ -42s & 50s+2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2}$$

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